The Theory of Computation
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Solution Manual
Chapter 7, Additional Problem 9

We prove this problem NP-complete with several different constructions, by transformation from *3-Dimensional Matching* or *X3C*.

1. Our first transformation (from *3DM*) creates a piece for each element, but with one connecting point for each occurrence of the element within a triple. Specifically, an element that appears in only one triple is transformed into a single vertex with that label; for each additional occurrence within a triple, we add three more vertices, one of which is labelled by the element name also (an attaching point); the other two are connected by an edge and, in addition, one of the two is connected to all attaching points for the element (meaning all such points eventually defined, not just those existing so far in the construction). (Actually, it is sufficient to connect in series: attachment–edge–attachment–edge–attachment, as long as needed.) Edges connect the pieces in $w - x - y$ pattern, with the proviso that each attaching point is used exactly once (for pieces from W or Y) or twice (for pieces from X) for such edges. This transformation is clearly polynomial-time (quadratic). (Note that the same transformation works from *X3C*, as it does not rely on ordering: the 3 sets can be connected as stated or in a triangle.)
   Since one of the two dummy vertices is connected to just one other vertex (the second dummy, that is connected to all attaching points for that piece), both dummy vertices must be included in an element of the partition, together with one of the attaching points (the only candidates for the third vertex). Thus only one attaching point of each piece is available for forming an element of the partition with vertices from other pieces; since the only edges left are the $w - x - y$ edges, such elements of the partition must be composed of a vertex from W, one from X, and one from Y. Thus we have a solution to *3DM* if and only if we have one from *PPL2*.
2. A second, simpler construction (also from *3DM*) sets up an edge for each element (with one endpoint designated for connections) and a

3-vertex, 2-edge piece for each triple, with the obvious connections (3 edges) between the triples' pieces and the elements' vertices. Since the other endpoint of each element's edge is of degree 1, any partition can use only one edge out of each element's connection point. The triple can be partitioned in only two ways, which finishes our proof.

3. Using one vertex per element of the set, and adding one connection piece for each triple, we can devise a number of transformations. One just uses an "E" (with 6 vertices and 5 edges, plus 3 connecting edges) and ends up looking quite similar to the previous construction; another uses triangles.
4. From *X3C*, a variety of transformations is also possible. Easiest is to set up one triangle for each 3-set, one segment for each element, and use one segment endpoint as connecting point, joined to all relevant triangle vertices by edges; in the solution, the chosen 3-sets correspond to triangles completely broken apart, while 3-sets discarded correspond to triangles forming a *PPL2* of their own. The E-shaped piece also works (with 9 vertices total, having vertices of degree 1 hanging from each branch of the E). A subtler trick, similar to the 1st transformation from *3DM*, uses 1 vertex for each element, and add extra pieces for extra connection points as needed; each addition consists of 3 vertices, say a_i, b_i, and v_i, with a_i connected to b_i, v_i, and the original v, and with v_i serving as new connection point.