The Theory of Computation

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Solution Manual
Chapter 7, Additional Problem 23

Membership in NP is non-trivial. The number of s–t paths in a digraph of n vertices can be exponential in n, so we cannot verify path by path that a given collection of arcs has the desired property. Note, however, that the collection as a whole has certain properties that allow verification. First note that we can eliminate any vertices and arcs that are not present on any path from s to t in linear time, thereby cleaning up the input. Now, if we were to remove every arc of the selected collection from the cleaned-up digraph, then we should observe that the resulting digraph has at most one path from any vertex to any other vertex—something that we can easily check in linear time. This property is clearly sufficient, so that the problem is in NP.

To prove the problem NP-complete, we reduce *Vertex Cover* to it. We assume that instances of *VC* do not include isolated vertices (or, equivalently, we discard such vertices if they exist). Given an instance of *VC*, i.e., a graph, $G = (V, E)$, and a bound, K, we create the following instance of our problem. The set of vertices of the digraph will be $V \cup E \cup \{s, t\}$, i.e., there will be a vertex for each vertex of the original graph, a vertex for each edge, and additional vertices s and t. We connect s to every vertex in V by an arc and connect every vertex in E to t by an arc; for each edge $e = \{u, v\}$ in the original graph, we place arcs (u, e) and (v, e) in the new digraph; and we also add the arc (s, t). The bound for our new problem is set to $|E| + K$.

Note that every path from s to t in our digraph (except for the single arc (s, t) itself) has length 3 and is of the form $s \to v \to e \to t$, where v is an endpoint of edge e in the instance of *VC*. Every vertex e in the digraph has in-degree two (from the two vertices that formed the endpoints of the corresponding edge in *VC*). Note that the number of s–t paths in our digraph is exactly $2|E| + 1$.

Given a solution V' to *VC*, we select as a solution in our new problem all arcs (s, v) where $v \in V'$, plus all arcs (e, t), for a total of at most $K + |E|$ arcs. This set of arcs distinguishes any pair of paths, as can be seen from the following two observations: (i) it distinguishes between the $|E| + 1$ arcs that reach t; and (ii) for each vertex e, it distinguishes between the two ways of reaching it (one from each vertex corresponding to an endpoint of the edge e) since it corresponds to a cover.

Conversely, note that: (i) any solution to our problem must distinguish between the two ways of reaching any vertex e corresponding to an edge, which will require that, for each such vertex e, at least one of the edges leading to it be included; (ii) it must distinguish between the $|E| + 1$ paths leading to t from its immediate predecessors; and (iii) the two are independent. To satisfy (ii), we need to select at least $|E|$ arcs. To satisfy (i), then, we are left with just K arcs to select; since the K arcs must distinguish, for each vertex e corresponding to an edge $e = \{u, v\}$, whether the path passes through u or through v, at least one arc on one of these paths must be included—which corresponds to selecting at least one of the two vertices u and v, which corresponds to a vertex cover.