The Theory of Computation
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Solution Manual
Chapter 7, Additional Problem 22

We prove this problem NP-complete (it is clearly in NP) by transforming *HP* to it. Given an instance of *HP*, $G = (V, E)$, we produce an instance of *Consecutive Sets* where $\Sigma = E \cup V$, $\Sigma_i = \{\{v_i, v_j\} \mid \{v_i, v_j\} \in E\} \cup \{v_i\}$, for $1 \le i \le |V|$, and $K = 2 \cdot |E| + 1$. Such a construction is clearly doable in polynomial time.

Note that, for $i \neq j$, we have $\Sigma_i \not\subseteq \Sigma_j$; moreover, we have $\Sigma_i \cap \Sigma_j = \{\{v_i, v_j\}\}$ if and only if $\{v_i, v_j\} \in E$. Finally we note that $K = 2 \cdot |E| + 1$ is exactly $\sum_{i=1}^{|V|} |\Sigma_i| - (|V| - 1)$.

Now, if there is a Hamiltonian path in G, say $v_{\pi(1)}, \ldots, v_{\pi(|V|)}$ for some permutation π, we can construct a solution string w of length K as follows. The ith block will contain the symbols of $\Sigma_{\pi(i)}$; the last symbol written in block i is also the first symbol in block $i + 1$ and corresponds to the edge $\{v_{\pi(i)}, v_{\pi(i+1)}\}$. Thus the string has length $\sum_{i=1}^{|V|} |\Sigma_i| - (|V| - 1)$ (the last term to account for shared "edges") and hence has length exactly K. Conversely, if there is a string w of length K in which the elements of Σ_i, for $i = 1, \ldots, |V|$, appear consecutively, then there is a corresponding Hamiltonian path in G. Note that such a string has a block of length $|\Sigma_i|$ for each Σ_i, yet has overall length shorter by $|V| - 1$ than the sum of the blocks' lengths; thus any two consecutive blocks share one symbol. As observed above, this symbol can only correspond to an edge between the corresponding vertices. Hence the sequence of vertices in G which corresponds to the sequence of blocks in w is such that any two consecutive vertices are connected by an edge: it is a Hamiltonian path.