The Theory of Computation

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Solution Manual
Chapter 7, Additional Problem 19

The problem is clearly in NP. We prove it NP-complete by transforming *Positive 1in3SAT* to it. Let the *Positive 1in3SAT* instance have m clauses and n variables; we produce $n + 1$ vectors of m components each; one vector is the enforcer and all its entries are 1s; each remaining vector corresponds to a variable and has a component of 1 for each clause in which the variable appears and a component of 0 otherwise. Finally, we set the bound B to 4.

If the *Positive 1in3SAT* instance has a solution, then we obtain a solution to our transformed instance by placing the enforcer alone in C_3, the vectors corresponding to false variables in C_2, and the remaining vectors in C_1. Since each clause has 1 true and 2 false variables, C_1 contributes 1 and C_2 contributes 2 to the sum; C_3, of course, contributes 1 and we have the desired bound. For the converse, note that each partition piece contributes at least one and that, since the component sum is uniformly 4 across the whole vector set, the three pieces must contribute at least 4. Thus a solution contributes exactly 4, as 1+1+2. If the enforcer vector is by itself, then the other two pieces are the true variables (the piece that contributes 1) and the false (the piece that contributes 2); otherwise, any of the three pieces can be selected for the true literals (!).

Another, very concise and elegant transformation starts from *G3C*. Start by eliminating the vertices of degree zero. Now, for each vertex we set up a vector of $|E|$ components, with a 1 in each component corresponding to an edge incident upon the vertex. We set the bound to 3. Note that the bound can only be equalled, since each partition piece must contribute at least one. Thus each partition piece in a solution must contribute exactly one. This means that the two vectors corresponding to the two endpoints of an edge can never be in the same partition piece, thereby corresponding to a legal 3-coloring.