The Theory of Computation

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Solution Manual
Chapter 7, Additional Problem 12

That *Simple Rural Postman* is in NP is somewhat non-trivial as a circuit could include the same edge many times and thus be very long. We need to show that a yes-instance always has a short certificate, that is, a short circuit—that the circuit is bounded in length by K does not help, since K could be exponential in the size of the input. (Although our proof produces a graph where any solution circuit is simple and although a similar proof can designed which produces a graph where any solution circuit is in fact simple, this only shows NP-completeness of the special cases and NP-hardness of the general case.) Assume that the graph is biconnected (if not, run a biconnected analysis (in polynomial time) and consider each component at a time); at worst, $E' = E$ and we must traverse each edge at least once. From the articulation point (or from an arbitrary point), we need to return to it at most as many times as there are edges to be visited in the component; hence each edge can never be visited more than $|E|$ times, putting an upper limit on the size of the circuit of $|E|^2$. This proves membership in NP.

To prove that it is NP-complete, we transform *Hamiltonian circuit* to it. Given the graph for *HC*, say $G = (V, E)$, we augment it as follows. For each vertex in $v \in V$, we add two new vertices, v' and v'', and associated edges, $\{v, v'\}$, $\{v', v''\}$, and $\{v'', v\}$. These new edges form the subset of designated edges; the maximum number of edges is set to $4|V|$. Such a transformation is clearly polynomial-time. (In effect, what we have done is to add an "ear" to each vertex; we now will require the postman tour to visit the ears by travelling along a Hamiltonian circuit at the base of the ears.)

Now, if *HC* has a solution, then a solution to *RP* can be formed by using the Hamiltonian circuit and, at each vertex, traversing the corresponding ear. The result is a non-simple cycle which traverses $|V| + 3 \cdot |V| = 4|V|$ edges. Conversely, a solution to *RP* must yield a solution to *HC*, as follows. Since the *RP* cycle has no more than $4|V|$, it cannot traverse more than $|V|$ edges in E (because it must traverse the $3|V|$ designated edges anyway); since it must visit each vertex in V (in order to traverse the associated designated edges) as part of the cycle, it traverses exactly $|V|$ edges in E;

hence the *RP* cycle restricted to the edges of E is a Hamiltonian circuit for the original problem.