The Theory of Computation

Bernard M.E. Moret

Addison-Wesley-Longman, 1998

Solution Manual
Chapter 7, Additional Problem 11

We prove this problem NP-complete (it is obviously in NP) by transforming *3SAT* to it. Our transformation is based on very simple graph pieces: while Grundy numberings appear fairly complex, we appeal only to the simplest results.

Specifically, note that a 2-cycle has only two possible Grundy numberings, each of which labels one vertex with a 0 and the other with a 1; thus 2-cycles can serve as truth-setting components. Now, note that 3-cycles (or k-cycles, $k > 2$, for that matter) do not have a Grundy numbering; however, a 3-cycle, one vertex of which is connected (by an arc issuing from the vertex) to a fourth vertex, has a Grundy numbering if and only if the fourth vertex is labelled with a 0. This gives us the solution: each clause will be represented by a 3-cycle, one vertex of which will be connected to the three corresponding "literals" in the truth-setting components; a vertex labelled 0 in the truth-setting component corresponds to a true literal. The rest follows easily.

From *NAE3SAT*, a similar transformation works, but now we connect each vertex in the 3-cycle to the corresponding literal vertex. A 3-cycle, each vertex of which is connected to one other vertex labelled 0 or 1, can be labelled if and only if the 3 endpoints are not all labelled 0 and not all labelled 1.