The Theory of Computation

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Solution Manual
Chapter 7, Problem 7.53

This result is often attributed to Kirchoff (who formalized the famous laws bearing his name that form the basis for circuit analysis). We proceed in two steps: first we show that we can compute the number of directed spanning trees of a directed Eulerian graph, then we use that result to compute the number of directed Eulerian circuits. The number of Eulerian circuits of an undirected graph is just half of the number of Eulerian circuits of the corresponding directed graph (where each edges has been replaced by a pair of arcs).

Define the Kirchoff matrix of a digraph $G = (V, A)$ to be a $|V| \times |V|$ matrix K where the (i, j)th entry is

$$K(i, j) = \begin{cases} \text{in-degree of vertex } i & i = j \\ 0 & i \neq j \text{ and } (i, j) \notin A \\ -1 & i \neq j \text{ and } (i, j) \in A \end{cases}$$

Note that the sum of the entries in a column of K is zero. We claim that, if every vertex of the graph has in-degree at most 1, then the graph contains a directed spanning rooted at vertex i if and only if the determinant of K_i (the matrix obtained by removing from K its ith row and column) equals 1. The "only if" part is simple. Assume that the graph has a spanning tree rooted at vertex i. Then the spanning tree accounts for all arcs of the graph, except possibly an arc into the root, since all vertices have in-degree at most 1. Thus the diagonal entry for vertex i is either 0 or 1 and all other diagonal entries equal 1; moreover, if we reorder the matrix according to a breadth-first search traversal of the spanning tree, the the (i, j)th entry must be 0 for $i > j$ and $i > 1$, so that the matrix K_i reordered in this fashion is upper-triangular with all diagonal elements equal to 1 and thus has determinant equal to 1. Now assume that no such spanning tree exists. If any vertex other than the ith has zero in-degree, its corresponding diagonal element is 0 and thus the determinant will be 0, as desired. Assume then that all other vertices have in-degree 1 and thus that all diagonal elements other than the

ith equal 1. Because every vertex (other than the ith) thus has a unique predecessor, we can trace back from an arbitrary vertex to its predecessor, its predecessor's predecessor, etc. Such tracing either reaches the i vertex or it does not. If it does, it is a branch of a partial spanning tree of G rooted at the ith vertex; thus there must be at least one case that it does not, but that can only happen if there is a cycle (that avoids the ith vertex). The sum of the columns corresponding to the vertices in that cycle is zero—that is, these columns are not linearly independent and thus the determinant is zero, as desired.

Now we can compute the number of distinct directed spanning trees of a digraph. The number of distinct spanning trees rooted at the ith vertex of an arbitrary digraph is simply the determinant of K_i. (This result is Kirchoff's main result, although he proved it in the case of directed multigraphs.) Since we know that determinants can be computed in time proportional to the cube of the dimension of the matrix, we can compute the total number of distinct directed spanning trees of a digraph on n vertices in $O(n^4)$ time.

Now we need to relate spanning trees and Eulerian circuits. The relationship was hinted at in the proof of Euler's theorem (Exercise 2.17). If we have a spanning tree rooted at some vertex v, we can build an Eulerian circuit as follows: start with any arc incident upon u and trace it back to its tail vertex; then continue tracing back, subject to not using the same arc twice and not using an arc from the spanning tree as long as another arc is available; stop when no unused arc remains. Since every vertex has equal in- and out-degrees, this process must terminate back at vertex v; we need only verify that it has used every arc of the digraph in the process. This is a simple proof by contradiction: if there was an unused arc (i, j), then there would have to be another unused arc (k, i), since the in-degree of i equals its out-degree; moreover that arc must be a tree arc, since such arcs are the last one used. Now we can repeat the construction starting from unused arc (k, i); eventually, we obtain an arc (v, w) that must be unused and thus an arc (t, v) that must be unused. But termination occurs at v when there is no unused arc incident upon v, a contradiction. Thus the circuit constructed is Eulerian.

From this procedure, we can count the number of Eulerian circuits in a digraph. For each directed spanning tree, we can construct one or more Eulerian circuits in this fashion. The number of choices has to do with the in-degree of each vertex: a vertex of in-degree k gives us $(k-1)!$ choices (not $k!$, because of our requirement that the tree arc be used last): $k-1$ ways out of it the first time, then $k-2$ the second time, and so on. Thus the number of choices for any fixed spanning tree is $\Pi_{v \in V}(\text{in-degree}(v) - 1)!$; note (the crucial observation) that this value is independent of the particular choice of spanning tree! Thus the total number of Eulerian circuits of a digraph is simply

$$ST(G) \cdot \Pi_{v \in V}(\text{in-degree}(v) - 1)!$$

where $ST(G)$ is the total number of directed spanning trees of G.

We have seen that we can compute $T(G)$ in $O(n^4)$ time; ignoring issues of number size, we can naively compute the product in time proportional to n^2; thus the complete computation takes $O(n^4)$ time and is polynomial.