The Theory of Computation
Bernard M.E. Moret
Addison-Wesley-Longman, 1998

Solution Manual
Chapter 7, Problem 7.45

Interestingly, the obvious symmetric counterpart of this statement is not known to hold: that is, we do not know whether *Unique Satisfiability* can be in coNP without forcing NP = coNP.

Assume that *Unique Satisfiability* is in NP. Then it has a concise certificate, which attests not only to the satisfiability of the instance, but also to the fact that it cannot be satisfied in more than one way. We want to show that *Unsatisfiability* is then in NP. Given an instance of *Unsatisfiability*, we transform it into an instance of *Unique Satisfiability* as follows. Denote the instance of *Unsatisfiability*, a Boolean expression over the variables $\{x_1, \ldots, x_n\}$, by E; set all variables to true and evaluate E, then evaluate E. If the result is true, we have a "no" instance of *Unsatisfiability* and map it to a fixed "no" instance of *Unique Satisfiability*, say the instance composed of one clause over two variables, $\{x_1, x_2\}$. Otherwise, set up $E' = E \cup \{\alpha, \overline{\alpha}\}$, where α does not appear in E. Then, if E has a satisfying truth assignment, E' has two, since the value of α is indifferent.

Otherwise, consider the new expression

$$(x_1 \Rightarrow E') \wedge (x_2 \Rightarrow E') \wedge \ldots \wedge (x_n \Rightarrow E') \wedge (\alpha \Rightarrow E')$$

Suppose that E cannot be satisfied; then this new expression can only be satisfied by setting every x_i and α to false (since then the implications are all of the type "false implies false," which is true). Thus the satisfying truth assignment for the new instance is unique. If E can be satisfied, then there are several ways in which we can satisfy the new expression: we can still assign "false" to all variables, but we can also assign a satisfying truth assignment for E—in which case every implication is either "false implies true" or "true implies true," both of which are true. Thus we have the desired result: the new expression is uniquely satisfiable if and only if the original is unsatisfiable. Note that, interestingly, the new expression is *always* satisfiable: the tricky part is the uniqueness.

The new expression can be constructed in polynomial time: each term in it is of the form $x \Rightarrow E$, or $\overline{x} \vee E$; the logical OR distributes over the clauses simply by adding the literal $\overline{x}$ inside each clause of E. Thus we effectively get $n + 1$ copies of the collection E, where each clause has been augmented by one literal. This is easily carried out in polynomial (at most quadratic) time.

Thus we have a many-one polynomial-time reduction from the coNP-complete problem *Unsatisfiability* to *Unique Satisfiability*. If the latter is in NP, it follows immediately from (the converse of) Exercise 7.9 that we must have NP = coNP.