The Theory of Computation

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Detailed Solutions
Chapter 7, Problem 7.41

We shall prove the circular chain of implications:

$$\text{NP} = \text{coNP} \Rightarrow \Delta_2^p = \text{NP} \cup \text{coNP} \Rightarrow D^p = \text{NP} \cup \text{coNP} \Rightarrow \text{NP} = \text{coNP}$$

Since $\text{NP} \cup \text{coNP} \subseteq D^p \subseteq \Delta_2^p$, the middle implication is immediate (D^p is "sandwiched" in the middle).

The first implication is fairly easy to prove, for instance as follows. Since we have NP = coNP, any problem in coNP must have a concise certificate—or, equivalently, both yes and no instances of any problem in NP must have concise certificates. Thus we can replace any call to an oracle for NP or for coNP by a certificate for the answer; in other words, a nondeterministic polynomial-time machine can do all the work by guessing the correct oracle answers, verifying them with the correct certificates, and otherwise doing the same work as the deterministic polynomial-time oracle machine. Thus the problem is in NP, i.e., $\Delta_2^p \subseteq \text{NP}$; since the containment holds the other way, the two classes are equal, which proves our first implication.

The last implication is the most difficult to prove. Perhaps the simplest way to go about it is to consider the known D^p-complete problem *SAT-UNSAT*; this problem consists of two collections of clauses and asks whether it is the case that the first collection is satisfiable while the second is not. If $D^p = \text{NP} \cup \text{coNP}$, then this problem is in NP or in coNP (or in both). If it is in both, we are done, since the problem is complete for D^p. If the problem is in NP (but not known to be in coNP), then it has a concise certificate; in particular, this means that the *UNSAT* part of the problem has a concise certificate and thus is in NP. But *UNSAT* is coNP-complete, so that we have NP = coNP. The symmetric reasoning holds if the problem is in coNP.