The Theory of Computation
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Solution Manual
Chapter 7, Problem 7.35

The problem is in P: we can grow S systematically by first finding the result of each operation on $S \times S$, then the result on each pair containing one or two of the newly created elements, and so on until no new element is created; then we verify whether w was among those created. The number of iterations is at most $|A| - |S|$ (since we must add at least one new element at each iteration) and the work at each iteration is no larger than $O(|A|^2)$, so that our algorithm is at worst $O(|A|^3)$.

To prove that the problem is complete, we transform PSA to it. Let the PSA problem be given by X, $S \subseteq X$, $T \subseteq X$, and $R \subseteq X \times X \times X$. We let the set of elements of our generation problem be $A = X \cup R \cup X \times R \cup \{\omega, \alpha\}$, where $\alpha, \omega \notin X$, with initial subset S. The element to be derived will be α (ω is an absorbing element) and the operation is defined by

$a \odot \omega = \omega \odot a = \omega$, for any $a \in A$
$a \odot \alpha = \alpha \odot a = a$, for any $a \in A$
$$a \odot a = \begin{cases} \alpha & a \in T \\ a & \text{otherwise} \end{cases}$$
$x \odot r = r \odot x = (x, r)$ for all $x \in X$ and $r \in R$
$(x, r) \odot (y, r) = z$ if $r = (x, y, z)$
all other compositions yield ω

(That operation is not generally associative.) The multiple "copies" of each element of X are there to allow unique definition of each operation even in the presence of several triples in R with the same first two entries. The entries from R serve as labels, with the operation doing the labelling $\odot\colon X \times R \to X \times R$; note however that the result of an operation which corresponds to a relation is put back in unlabelled form. It is easily seen that f can be derived from the initial subset if and only if some element of T is accessible.