The Theory of Computation

Bernard M.E. Moret

Addison-Wesley-Longman, 1998

Solution Manual
Chapter 7, Problem 7.32

Membership in NP is obvious. We transform *Vertex Cover* to our problem as follows. S will contain V and an additional $|V| - K$ elements, call them $a_1, \ldots, a_{|V|-K}$. We set up C to have two types of elements: the first type is such that only a set X corresponding to a vertex cover will not be contained in any of them; the second provides an enforcement of K by ensuring that any reasonable X will be contained in all of them. The first type has an element for each edge $\{v_i, v_j\}$; the element is the set $V - \{v_i, v_j\}$; the second type has elements $s_1, \ldots, s_{|V|-K}$, where $s_i = V \cup \{a_i\}$. (The only reason for the a_i is to make it possible to have $|V| - K$ "copies" of V in a set.) Now B is such that any set of n elements is contained in $|V| - n$ its elements: B has $|V|$ elements, with the ith element being $V - \{v_i\}$. (Note that B is set up this way to offer a threshold equal to that of C; note also that the threshold depends on K, which is the whole reason for this part: otherwise we might as well have let $B = \emptyset$ and C be composed only of the first type of element.) The whole construction is easily carried out in polynomial time.

Now, a solution to vertex cover easily yields one to our problem: let X be composed of the elements of the cover. By design, X is contained in each of the s_i in C and not in any of the other elements of C; thus it is contained in $|V| - K$ elements of C; since it contains $|X|$ elements, it is contained in $|V| - |X|$ elements of B; but $|X| \leq K$ by hypothesis, hence we have a solution. Conversely, note first that any solution X never contains any of the a_i elements; thus a solution X is in effect a subset of V. Now, X is automatically contained at least $|V| - K$ times in C (the s_i elements) and at least $|V| - |X|$ times in B. It could be contained more often in C, but we must have as a minimum condition $|V| - |X| \geq |V| - K$, hence $|X| \leq K$. Thus X must meet the size requirements for a cover. We claim that it does correspond to a vertex cover for the original graph. The problem is that we could very well get a solution set X with $|X| < K$ such that X is contained more often than $|V| - K$ times in C and thus does not correspond to a cover; but this is easily corrected: we'll create X' such that $X \subseteq X'$, $|X'| \leq K$, and X' corresponds to a cover. We simply take any element of C in which X is contained that indicates an edge not covered and add one of the vertices to

X; this operation decreases the number of containments in C by at least 1 (possibly more) and the number of containments in B by exactly 1; thus our larger X is still a solution. We repeat until the set obtained, X' is contained exactly $|V| - K$ times in C; by our earlier discussion, X' is then a solution. Since it is contained exactly $|V| - K$ times in C, it also corresponds to a cover.