The Theory of Computation

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Solution Manual
Chapter 7, Problem 7.26

We prove this problem NP-complete by transforming $X3C$ to it.

Given an instance S, $|S| = 3q$, $C \subset 2^S$, $(c \in C \Rightarrow |c| = 3)$ of $X3C$, we construct an instance $G = (V, E)$, $R \subset V$, $K \leq |V| - 1$ of STG as follows. We let $V = S \cup \{z\} \cup \{x_1, \ldots, x_{|C|}\}$ (where the x_is are intended to represent 3-sets), $R = S \cup \{z\}$, $E = \{\{a_i, x_i\}, \{b_i, x_i\}, \{c_i, x_i\} | \{a_i, b_i, c_i\} \in C\} \cup \{\{z, x_i\} | 1 \leq i \leq |C|\}$, and $K = 4q$. All edges have unit length. This transformation is clearly polynomial-time.

Now, given a solution to $X3C$, we obtain a solution to STG by including all corresponding x_is and their associated edges in our Steiner tree; the disjointness of the 3-sets in the solution guarantees that the result is a tree with exactly $4q$ edges. Conversely, observe that any solution to STG must have exactly $4q$ edges (at most that many are allowed, and at least that many are needed to produce a tree that includes all vertices from S as well as z). Thus in fact, exactly q edges emanating from z are included: those are the 3-sets in the solution for $X3C$; the tree structure guarantees disjointness.

An alternate construction omits the root and connects all 3-set vertices in a complete graph; $R = S$ and $K = 4q - 1$ and all edges still have unit length.

From VC, we can copy the graph, but complete it into the complete graph on $|V|$ vertices; now, for each original edge $e = \{u, v\}$, we set a new vertex e and associated edges $\{u, e\}$ and $\{v, e\}$. The bound is just $B + |E|$ (B is the bound of VC) and R consists of all the new vertices e. Since a tree of n vertices has $n - 1$ edges (a result we proved in Chapter 2) and since our edges all have the same length, we can simply look at the minimum number of vertices needed in the Steiner tree: it is clearly all of the new vertices (the set R of size $|E|$), plus enough additional vertices to turn the whole into a tree. Since the new vertices corresponds to edges of the old graph and are only connected to vertices of the original graph, any Steiner tree must include one or both of $\{u, e\}$ and $\{v, e\}$, i.e., a vertex cover of the original graph; these vertices can then be connected into a tree through a simple chain in the complete graph.