The Theory of Computation

Bernard M.E. Moret

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Solution Manual
Chapter 7, Problem 7.2

Note that *VC* is clearly in NP, as it suffices to guess a cover of size obeying the bound and then to scan all edges of the graph, verifying that each is covered by some vertex in the cover.

The construction is along the same lines as for *MaxCut*, but starts from *3SAT*, connects vertices in the triangles to the corresponding vertices in the segments devoted to truth assignment (rather than to those corresponding to complements), and requires slightly different arguments. Note that, in order to cover all three edges of a triangle, at least 2 of the 3 vertices of the triangle must be selected as part of the cover. Thus we shall set the size the desired cover at $n + 2k$ (the number of variables plus twice the number of clauses). The whole transformation easily runs in low polynomial time.

Given a solution to *3SAT*, we shall place in the cover all vertices in the truth-setting component that correspond to true literals; these n vertices cover the truth-setting edges, as well as the edges connecting the truth-setting components to the true literals in each clause. For each triangle, select one of its vertices corresponding true literals (there may or may not be a choice) and place the other two in the cover. Note that, as observed, all edges leading to triangle vertices that correspond to true literals are already covered; selecting the other two vertices covers the triangle and any other edges connecting to the triangle. Thus we have a solution of size $n + 2k$ to the cover problem.

Given a vertex cover of size $n + 2k$, first note that, as mentioned earlier, at least $2k$ of these vertices are needed just to cover the triangles; since the truth-setting edges cannot be covered by these vertices, they use up all remaining n vertices, one for each edge. Thus the structure of the vertex cover is well-defined: one vertex chosen out of each pair used in truth-setting for the variables and two vertices chosen out of three in each triangle. Note that the edges leading from the truth-setting components to the third vertex of each triangle (the one not in the cover) must then be covered by the selected truth-setting vertex. To obtain a satisfying truth assignment, we set to True the literals that correspond to the selected truth-setting vertices; this ensures a legal truth assignment since exactly one vertex of each pair is selected and also ensures a satisfying truth assignment, since the one vertex not selected in the the cover of each triangle always corresponds to a true literal.