The Theory of Computation

Bernard M.E. Moret

Addison-Wesley-Longman, 1998

Solution Manual

Chapter 7, Problem 7.13

Any dynamic program that returns the optimal solution to a problem can be trivially made to count the number of optimal solutions: in addition to storing the optimal value at each step (in each table location), the program also stores the number of ways in which that value can be obtained. When computing the next optimal value, the program can combine the number of ways in which each of the pieces can be optimize to compute the number of ways in which the new, larger piece can be optimized. (Most of the time, this computation is a simple multiplication.)

So we need to write down the dynamic program recurrence for matrix chain products. The problem asks for the least expensive way to parenthesize (to associate) a product of n rectangular matrices. We can write an expression to find the best way to parenthesize the subexpression including operands i through j ($i \leq j$):

$$min_cost(i, i) = 0$$

and, for $i < j$,

$$min_cost(i, j) = \min_{k=i,i+1,\ldots,j-1} (min_cost(i, k) + min_cost(k + 1, j) + cost(i, k, k + 1, j))$$

where the function $cost(i, k, k + 1, j)$ returns the cost of multiplying the matrix formed by the product of matrices i through k with the matrix formed by the product of matrices $k + 1$ through j. Since the cost of multiplying an $l \times m$ matrix by an $m \times n$ matrix is of order $O(lmn)$, the function $cost(i, k, k + 1, j)$ just returns the product of the number of rows of matrix i, the number of columns of matrix k, and the number of columns of matrix j. Our dynamic program will also store the sum, over all values of k that achieve the minimum cost for $min_cost(i, j)$, of the products of the number of ways in which $min_cost(i, k)$ could be achieved and of the number of ways in which $min_cost(k + 1, j)$ could be achieved; this sum is the number of ways in which $min_cost(i, j)$ can be achieved.

Thus the dynamic program can compute the number of distinct solutions at no additional (asymptotic) expense over its computation of the solution matrix.