The Theory of Computation

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Solution Manual
Chapter 6, Problem 6.6

What we need to prove is that, if some problem A is in P and some problem B many-one reduces to A in polynomial time, then B is also in P; the we should repeat for Exp; finally we need to show that such need not be the case for E.

We begin with P. Since A is in P, there is a Turing machine M and a polynomial p such that M can decide an instance x of A in time $p(|x|)$. Let f be the polynomial-time transformation from B to A and let q be its polynomial bound. We build a decision machine M' for B by using the machine M for A and the the transformation f. Specifically, M' will decide an instance x of B by transforming it in time $q(|x|)$ to an instance $y = f(x)$ of A and running the decision procedure M on y in $p(|y|)$ time. Since the transformation takes time $q(|x|)$, what it produces, the string y, cannot exceed $q(|x|)$ in length; thus the time taken by solver M, which is $p(|y|)$, is at most $p(q(|x|))$, which is a polynomial in $|x|$. Hence our solver M' runs in time polynomial in $|x|$ and thus B is in P.

The same reasoning applies if we replace the polynomial bound p by an exponential bound of the type 2^{n^k} for some $k \in \mathbb{N}$, since we can bound any quantity of the form $2^{q^k(n)}$ by another exponential of the form 2^{n^m} for a suitable choice of m.

However, if the bound is a bound for E, that is a bound of the form 2^{kn}, then the running time after the transformation is of the form $2^{kq(n)}$, which, if q is a polynomial of degree higher than 1, cannot be bounded by any function of the form 2^{mn} for any constant m. Thus E need not be closed under polynomial-time transformations. To prove that it is actually not closed, we need a specific counterexample; that is much harder, since it requires knowledge of specific problems with specific lower bounds—we need to know of at least one problem requiring more time than is available in E and that happens to reduce to some problem in E in polynomial time.