The Theory of Computation

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Solution Manual
Chapter 6, Problem 6.18

It is clear that, if we can prove the implication in the hint, we get the desired contradiction, since we would obtain PSPACE $\subseteq$ DSPACE(n), whereas the hierarchy theorem for space implies DSPACE(n) $\subset$ PSPACE, with strict containment. To prove the desired implication, we take a string of an arbitrary problem in PSPACE, pad it to make it look like it belongs to a problem in DSPACE(n), which places it also in P, then show that the padding does not affect the solvability in polynomial time.

So let Π be a problem in PSPACE; if it is in PSPACE, it is because it is in DSPACE(k) for some fixed k. Let $x \in \Pi$ be an arbitrary instance of Π; consider the new string $y = x\#0^{|x|^{k-1}}$, where the character # is one never used by the known DSPACE(k) algorithm that solves Π. That algorithm solves instance x in $O(|x|^k)$ space. But it (well, a very slight modification thereof) also solves y in $O(|y|)$ space—the character # acts as a separator beyond which the modified algorithm never reads; thus the input y has length $|y| = |x|^k + 1$, but the algorithm only "sees" the first $|x|$ bits of it. So, the set of all ys derived in this fashion from all instances x of Π forms a problem of its own, call it Π'. We have just shown that Π' is in DSPACE(n); thus, by our hypothesis, it is also in P. Thus it remains to prove that, if we can solve Π' in polynomial time, we can also solve Π in polynomial time. If there exists an algorithm that solves an instance y of Π' in $p(|y|)$ time for some polynomial p, then an instance x of Π can be solved as follows: first pad the instance as described above to form $y = x\#0^{|x|^k}$, then solve the y with the known polynomial-time algorithm. The first step (padding) takes $O(|x|^k)$ time, while the second (solving) takes time $O(p(|y|)) = O(p(|x|^k))$; thus both steps run in time polynomial in $|x|$ and thus x can be solved in polynomial time. Our conclusion follows.