The Theory of Computation

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Solution Manual
Chapter 6, Problem 6.1

We shall map the elements of the set in $X2C$ to vertices of the graph in *Matching* and 2-sets of $X2C$ to edges of the graph in *Matching*. Specifically, given an instance of $X2C$ as the set S (where $|S|$ is even) and the collection $\mathcal{C}$ of 2-subsets of S, we create an instance $G = (V, E)$ of *Matching* by setting $V = S$ and $E = \mathcal{C}$. We claim that the instance of $X2C$ has a solution (an exact cover) if and only if the instance of *Matching* has a matching of size $|S|/2$.

Assume then that we have an exact cover for S, that is, a collection of $|S|/2$ of 2-subsets of S, the union of which is S. Then these subsets, viewed as edges of G, share no endpoint (because they are pairwise disjoint) and thus form a matching of the required size. Conversely, assume that we have a matching for G of size $|S|/2$, that is, we have $|S|/2$ edges with no shared endpoint. Each edge correspond to a 2-subset of $X2C$; since these edges share no endpoint, their corresponding subsets are all disjoint; but then they form a cover, since together they cover $2 \cdot |S|/2 = |S|$ elements, i.e., the entire set.

The reduction takes linear time, since it basically does no more than copy the input.