The Theory of Computation
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Solution Manual
Chapter 2, Additional Problem 2.2

1. If $(k+1)^n$ was $O(k^n)$, we would have constants c and N such that, for all $n > N$, $(k+1)^n \leq c \cdot k^n$; but, after dividing both sides by k^n (a positive, non-zero value), this becomes $((k+1)/k)^n \leq c$ or $c \geq (1+\frac{1}{k})^n$, which cannot be true for all $n > N$, since the right-hand side of the inequality grows without bounds as n grows.
   The second statement can be rewritten, after simplification, as "$n \log(k+1)$ is $O(n \log k)$." That is pretty obviously true, since both terms are $\Theta(n)$—both are just n times a constant factor. Formally, we would write $n \log(k+1) \leq c \cdot n \log k$ for all $n \geq N$, yielding $c \geq \log(k+1)/\log k$, certainly an inequality that can be met for all values of n. So we can set $c = \log(k+1)/\log k$ and $N = 1$.
2. While it is fastest to solve this problem by using Sterling's approximation for factorials, it is most instructive to solve it without. Let us start with Sterling's approximation:

$$(\log n)! \approx \sqrt{2\pi \log n} \left(\frac{\log n}{e}\right)^{\log n}$$

   Using the approximation and setting up the inequality, we get, if the first function is to be $O(\,)$ of the second,

$$n^{\log\log\log n} \leq c \cdot \sqrt{2\pi \log n} \left(\frac{\log n}{e}\right)^{\log n}$$

   Taking logs on both sides, we get

$$\log\log\log n \log n \leq \log c + 0.5 \log 2\pi + 0.5 \log\log n + (\log\log n - \log e) \log n$$

   or

$$\log c \geq \log\log\log n \log n - 0.5 \log 2\pi - 0.5 \log\log n - (\log\log n - \log e) \log n$$

$$= (-\log\log n + \log\log\log n + \log e)\log n - 0.5\log\log n - 0.5\log 2\pi$$

Asymptotically, the right-hand side is dominated by the $-\log\log n \log n$ term and so decreases without bound, so that the inequality is respected for all n beyond some N. We conclude that $n^{\log\log\log n}$ is $O\big((\log n)!\big)$. By reversing the inequality, we can also immediately verify that $n^{\log\log\log n}$ is not $\Omega\big((\log n)!\big)$.
Without Sterling's approximation, we can prove the $O(\,)$ relationship as follows. We want to show that there exist positive constants c and N such that, for all $n \geq N$, $n^{\log\log\log n} \leq c \cdot (\log n)!$. Taking logs on both sides, we get

$$\log\log\log n \log n \leq \log c + \sum_{i=1}^{\log n} \log i$$

Notice that the last $\frac{1}{2}\log n$ terms of the sum are all at least as large as $\log(\frac{1}{2}\log n)$, so that the sum is at least as large as $\frac{1}{2}\log n \log(\frac{1}{2}\log n)$. Conversely, none of the terms in the sum exceeds $\log\log n$, so that the sum does not exceed $\log n \log\log n$; therefore the sum is $\Theta(\log\log n \log n)$. Now, that means that the right-hand side of the inequality grows faster than the left-hand side, so that, for each choice of $c > 0$, the inequality is respected for all n beyond some suitable N_c. Again, inverting the inequality shows that there is no constant c that can work in the reverse direction.