The Theory of Computation

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Solution Manual
Chapter 2, Problem 2.9

The relationships are: (2) is a subproblem of (1) and (4) is (a version of) a subproblem of each of the other three.

That (2) is a subproblem of (1) is clear: the two problems have exactly the same question, but the set of legal instances of (2) is a subset of the set of legal instances of (1).

Problem (3) asks a different question and it is not hard to find instances common to (1) and (3) or to (2) and (3) that yield one answer for (1) or (2) and a different answer for (3). For instance, consider the set $\{a, b, c, d, e\}$ with subsets $\{a, b, c\}$, $\{a, b, e\}$, $\{a, c, d\}$, and $\{c, d, e\}$. This is a legal instance of (1), (2), and (3) (but not of (4)); in both (1) and (2), a solution is the pair of subsets $\{a, b, c\}$ and $\{c, d, e\}$, but in (3) there is no solution at all! Thus (3) is not a subproblem of either (1) or (2), nor is either (1) or (2) a subset of (3).

Finally, consider (4): we claim that it is a subproblem of (1), (2), and (3). Note that its legal set of instances is a restricted subset of the legal set of instances of any of the other three—we further restrict the instances of (2) by requiring a certain size for the set itself (a multiple of 3). Now note that any cover for a set of $3n$ elements made entirely of subsets of 3 elements each must have at least n subsets; hence asking for n subsets is asking for the smallest possible cover. In that sense, the problem is a subproblem of (1) and (2), although there remains a difference: if no cover of size n exists, (1) and (2) will still return a (larger) cover, but (4) will simply return "no solution." Requiring that the cover consists of n subsets of 3 elements each means that each of the $3n$ elements of the set appears in exactly one of the n subsets—otherwise we would not have a cover with just n subsets. Thus the requirement that the cover have size n is equivalent to a requirement that the subsets forming the cover be disjoint; thus (4) effectively asks exactly the same question as (3), so that it is a subproblem of (3).