# The Theory of Computation

Bernard M.E. Moret

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## *Solution Manual*
## *Chapter 2, Problem 2.7*

1. By induction on $n$. By definition $\langle 0, 0\rangle$ must be the smallest value produced by a pairing function (because of monotonicity), thus it must be 0. Assume then that $\langle 0, 0, \ldots, 0\rangle_{n-1}$ equals 0; by definition, we have

$$\langle 0, 0, \ldots, 0\rangle_n = \langle 0, 0, \ldots, \langle 0, 0\rangle\rangle_{n-1} = \langle 0, 0, \ldots, 0\rangle_{n-1} = 0$$

The pairing of $x$ with zeros is quite similar. As basis, we have

$$\langle x, 0, 0\rangle_3 = \langle a, \langle 0, 0\rangle\rangle = \langle x, 0\rangle$$

(the last from the result above) and the step gives us

$$\langle x, 0, 0, \ldots, 0\rangle_n = \langle x, 0, \ldots, \langle 0, 0\rangle\rangle_{n-1} = \langle x, 0, \ldots, 0\rangle_{n-1} = \langle x, 0\rangle$$

as desired.

2. By definition, we have $\Pi_i^{n+1}(z) = \Pi_i^n(z)$ for $i < n$; thus we have $\Pi_0^n(z) = \Pi_0^{n-1}(z) = \ldots = \Pi_0^1(z)$. Again by definition we have $\Pi_i^1(z) = z$ for all $i$. Thus we have $\Pi_0^n(z) = \Pi_0^1(z) = z = \Pi_1^1(z)$..
By definition, we have $\Pi_i^{n+1}(z) = \Pi_2(\Pi_n^n(z))$ for all $i > n$; thus, in particular, we obtain the same value for $i = n + 1$ and for any larger $i$, so we can write $\Pi_{n+1}^{n+1}(z) = \Pi_i^{n+1}(z)$ for all $i \geq n + 1$, or, reducing $n$ by 1, $\Pi_n^n(z) = \Pi_i^n(z)$ for all $i \geq n$.

3. We have, by definition $\Pi_1^0(z) = 0$, which is obviously no larger than any $z$. Similarly. we have $\Pi_1^1(z) = z$, again no larger than $z$. These two form the basis of an inductive proof. Assume then that we have $\Pi_1^j(z) \leq z$ for all $j$ up to $n - 1$ and consider $\Pi_1^n(z)$. By definition, we have $\Pi_1^n(z) = \Pi_1^{n-1}(z)$; but the second term is no larger than $z$ by inductive hypothesis, so our conclusion holds.