The Theory of Computation

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Addison-Wesley-Longman, 1998

Solution Manual
Chapter 2, Problem 2.6

The first thing to note about our definition of the projections functions is that the recursion terminates. The index of recursion is the superscript (the j in Π_i^j). We have three families of base cases, Π_i^0, Π_i^1, and Π_i^2, and a recursive step, Π_i^j for $j > 2$. The recursive step always reduces the superscript by one, so termination is assured.

It thus remains to show that our definition properly reverses pairing; again, as in the statement of the exercise, note that the projection function may return any value whatsoever when its indices are not compatible with the reversal of a pairing—given Π_i^j, we need only worry about the range $1 \leq i \leq j$. Moreover, since the base cases are all clearly correct (the only interesting case is for $j = 2$), we can also assume $j > 2$. So let $z = \langle x_1, \ldots, x_{n-2}, x_{n-1}, x_n \rangle_n$ for some $n > 2$ (thus x_1 and x_{n-2} might be the same entry); we want to prove that we have $\Pi_i^n(z) = x_i$ for $1 \leq i \leq n$. Our proof is by induction: values of n up to 2 give us the basis and we now want to show that the statement hold for some $n > 2$ if it holds for $n - 1$ and all lower values. By definition of pairing, we have

$$z = \langle x_1, \ldots, x_{n-2}, x_{n-1}, x_n \rangle_n = \langle x_1, \ldots, x_{n-2}, \langle x_{n-1}, x_n \rangle \rangle_{n-1}$$

and so we can write

$$z = \langle y_1, \ldots, y_{n-2}, y_{n-1} \rangle_{n-1}$$

where we have set $y_k = x_k$ for $1 \leq k \leq n - 2$ and $y_{n-1} = \langle x_{n-1}, x_n \rangle$. By definition of our projection functions, we have (for $n > 2$)

$$\Pi_i^n(z) = \begin{cases} \Pi_i^{n-1}(z) & i < n - 1 \\ \Pi_1(\Pi_{n-1}^{n-1}(z)) & i = n - 1 \\ \Pi_2(\Pi_{n-1}^{n-1}(z)) & i > n - 1 \end{cases}$$

By inductive hypothesis, all of the Pi_k^{n-1} return the kth of $n - 1$ paired values, so we have $\Pi_k^{n-1}(z) = y_k$ for $1 \leq k \leq n - 1$.

In the third case, we must have $i = n$ and so we can write

$$\Pi^n_n(z) = \Pi_2(\Pi^{n-1}_{n-1}(z)) = \Pi_2(y_{n-1}) = \Pi_2(\langle x_{n-1}, x_n\rangle) = x_n$$

as desired. In the second case, we have

$$\Pi^n_{n-1}(z) = \Pi_1(\Pi^{n-1}_{n-1}(z)) = \Pi_1(y_{n-1}) = \Pi_1(\langle x_{n-1}, x_n\rangle) = x_{n-1}$$

as desired. Finally, in the first case ($i < n - 1$), we have

$$\Pi^n_i(z) = \Pi^{n-1}_i(z) = y_i = x_i$$

(by inductive hypothesis) as desired.