The Theory of Computation

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Solution Manual

Chapter 2, Problem 2.5

By inspection, this function is monotonic in both x and y (all terms are positive). To verify that it is a pairing function, we need to verify injectivity and surjectivity. As in Exercise 2.4, we shall verify surjectivity by showing that we can invert the pairing.

We thus begin by showing injectivity. The key to injectivity is to realize that the function creates a perfect square, then adds x to it; thus, intuitively, the function is injective because, between two consecutive perfect squares, x determines the value of the function—and then, knowing x, the value of y is also fixed. Formally, we begin by verifying that x cannot be large enough to bridge the difference between two consecutive perfect squares. An increase of 1 in the term being squared causes an increase of $2y + 2\lfloor \frac{x+1}{2} \rfloor + 1$ in the value of the function; this increase is indeed always at least as large as x, so that x cannot bridge the difference between consecutive perfect squares. That being the case, the function is injective: for two pairings to yield the same result, they must have the same perfect square and thus the same x; but then they have the same y.o

The observation about perfect squares also show that the function is easily reversed: find the largest perfect square that does not exceed the value of the function; the difference is the value of x. Then take the root of the perfect square and subtract the floor of $\frac{x+1}{2}$ to obtain y. Thus the inversion given in the problem is correct and the function is surjective. Hence we have a pairing function.