The Theory of Computation
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Addison-Wesley-Longman, 1998

Solution Manual
Chapter 2, Problem 2.42

Let S and T be two countable sets; that is, we have a bijection $f: \mathbb{N} \longleftrightarrow S$ and another $g: \mathbb{N} \longleftrightarrow T$.

Union The union $S \cup T$ can be enumerated as follows: define $h\colon S \cup T \longrightarrow \mathbb{N}$ by

$$h(z) = \begin{cases} 2f(z) & z \in S \\ 2g(z) + 1 & \text{otherwise} \end{cases}$$

It is clear that h is injective: both f and g are injective and we use f and g on entirely disjoint sets to produce entirely disjoint sets of values. Thus the cardinality of $S \cup T$ cannot exceed that of $\mathbb{N}$ and thus $S \cup T$ is countable. (Producing a bijection is rather harder, unless, of course, S and T are disjoint; but an injection suffices to prove the point.)

Intersection The intersection $S \cap T$ can be enumerated as follows: define $h\colon S \cap T \longrightarrow \mathbb{N}$ by $h(z) = f(z)$. Since f is injective, so is h and we are done. (Again, defining a bijection—when $S \cap T$ is not finite, that is—is harder.)

Cartesian Product The Cartesian product $S \times T$ can be enumerated as follows: define $h\colon S \times T \longrightarrow \mathbb{N}$ by dovetailing on $S \cup T$ exactly as we did for the rational numbers, using $f(x)$ and $g(y)$ (for $x \in S$ and $y \in T$) as indices:

$$h(x, y) = \frac{1}{2}(f^2(x) + g^2(y) + 2f(x)g(y) - f(x) - 3g(y) + 2)$$

Because both f and g are bijections and because our dovetailing process induces a bijection, h is also a bijection.