The Theory of Computation

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Solution Manual
Chapter 2, Problem 2.36

This question has two parts, the second not well defined. We shall use only positive rationals—adding the negative rationals does not really change the problem. It is rather easy to establish a bijection between $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{Q}$, at least if we view $\mathbb{Q}$ as formal fractions (i.e., if we consider two fractions of the same value, but with different numerators and denominators, to be distinct). For instance, we can simply pair up the the two numerators and pair up the two denominators, considering the two numbers thus produced to be the numerator and denominator of the new fraction. This mapping can clearly be inverted using the projection functions corresponding to the pairings used, so we have a bijection.

The interesting part comes when considering the requirement of monotonicity. Obviously, we can no longer define monotonicity in quite the same way, but we could ask that an increase in either argument cause an increase in the result. The construction used above, i.e.,

$$f : (\frac{a}{b}, \frac{c}{d}) \rightarrow \frac{\langle a, c\rangle}{\langle b, d\rangle}$$

may no longer be monotonic in each argument: it is possible to increase the fraction $\frac{a}{b}$ by increasing *both* a and b, in which case we do not really know how much each of $\langle a, c\rangle$ and $\langle b, d\rangle$ increases, so we cannot say much about the behavior of the resulting fraction.

On the other hand, note that "increasing in value" may not be the right criterion when we consider fractions to be formal symbols. In that case, however, what is a valid definition of monotonicity? We need an ordering on $\mathbb{Q}$ to define it; with an ordering, the definition can be the same as for $\mathbb{N}$: as we move from one element of $\mathbb{Q}$ to the next in the argument pair, the result of the bijection must itself move from an element of $\mathbb{Q}$ to one farther down in the ordering. Any pairing from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ gives us a possible ordering for $\mathbb{Q}$. Once we have this ordering, think of replacing each fraction in $\mathbb{Q}$ by its index in the ordering: thus fraction $\frac{a}{b}$ has some index i and fraction $\frac{c}{d}$ has some index j; a pairing function applied to i and j will give us the new natural number $k = \langle i, j \rangle$, which is larger than either i or j; but k also corresponds to a fraction and, by definition, to a fraction that is farther in the ordering than either $\frac{a}{b}$ or $\frac{c}{d}$. If we now replace $\frac{a}{b}$ by $\frac{a'}{b'}$ which is farther in the ordering—say it has index $i + n$ for some n—, then our bijection now produces the fraction corresponding to the index $\langle i + n, j \rangle$, which is larger than k. Thus increasing the first argument in the bijection increases the result. The same is obviously true of the second argument. Thus we have a bijection that is also monotonic in each argument—a pairing function between $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{Q}$.