The Theory of Computation

Bernard M.E. Moret

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Solution Manual

Chapter 2, Problem 2.31

Let S have m elements and T have n elements and consider functions from S to T.

- Assume that we have $m \geq n$. How many distinct surjective functions are there? Because we must map n of the elements of S to the n elements of T, we get $\binom{m}{n} \cdot n! = m!/(m-n)!$ choices there (how to choose the n elements and then the obvious $n!$ permutations of mapping). The remaining $m-n$ elements of S can each be mapped onto anything at all in T, and so appear to give another factor of n^{m-n}, for a total of $n^{m-n}m!/(m-n)!$ choices. However, this expression overcounts the number of choices: the n^{m-n} remaining choices in fact will cause repetitions, even when they are all mapped to the same element of T, because one or more of these $n-m$ mappings could be used in place of the n mappings chosen earlier. For example, consider mapping the set $\{a, b, c, d\}$ to the set $\{0, 1\}$. We can pick the subset $\{a, b\}$ to map injectively to $\{0, 1\}$, say with a mapped to 0 and b mapped to 1, then pick any mapping for c and d, say by mapping c to 0 and d to 1; but that is indistinguishable from picking the subset b, c to map injectively to $\{0, 1\}$ and then mapping a to 0 and d to 1.
  To come up with an exact formula, we need to use the inclusion/exclusion principle: start with the $m!/(m-n)!$ choices for the injection on a subset of n elements, then painstakingly count the number of choices for the remaining elements. We will not obtain a nice and concise formula from this process: instead, it will remain in a sum form.
- Assume that we have $m \leq n$. How many distinct injective functions are there? That one is quite simple: we have n choices for mapping the first element of S, then $n-1$ for the second, and so on; thus we have $n!/(n-m)!$ choices in all.