# The Theory of Computation
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*Solution Manual*
*Chapter 2, Problem 2.28*

Since we need to prove an equality, we have two parts to the proof. Let $G = (\{U, V\}, E)$ be our bipartite graph—we need not have $|U| = |V|$.

One part is very easy. Let $M \subseteq E$ be a matching of maximal size for $G$. We want to show that a vertex cover of minimum size for $G$ must have size at least $|M|$. Since no two edges in $M$ share a vertex, a vertex cover for just $M$ will require at least one vertex per edge and thus at least $|M|$ vertices.

Let $C \subset U \cup V$ be a vertex cover of minimum size for $G$. We want to show that $G$ has a maximum matching of size at least $|C|$. If no edge of $G$ has both endpoints in the cover, the problem is trivial: for each cover vertex, just pick one of the edges that it covers. Otherwise there must be at least one cover vertex that covers only edges that have no other endpoint in the minimum cover. (The only graph for which that is not true is the complete graph on$n$ vertices, which is not bipartite.) For that vertex, we pick an arbitrary edge among those it covers for the matching, then remove the matched edge and all edges that share an endpoint with it from the graph. The result is a subgraph with two fewer vertices and at least one less edge, which has a minimum cover of size $|C| - 1$. By induction, that subgraph has a matching of size at least $|C| - 1$; to this matching we can add the matched edge we had picked, since it cannot share any endpoint with edges from the subgraph. Thus we have obtained a matching of size $|C|$ for $G$. (The basis for the induction is simple: the statement is true for graphs with no vertex on one side of the bipartite graph, and thus no edge at all.)

Taken together, our two results show that the size of a maximum matching is bounded above and below by that of a minimum cover, so that the two must be equal.