# The Theory of Computation
Bernard M.E. Moret
Addison-Wesley-Longman, 1998

## *Solution Manual*
## *Chapter 2, Problem 2.27*

This is the proof of Hall's theorem. Since the theorem states necessary and sufficient conditions, our proof has two parts. Let the bipartite graph be $G = (\{U, V\}, E)$, with $|V| < |U|$.

The necessary condition is easy. If $G$ has a matching of size $|V|$, denote that matching by the assignment of each vertex of $V$ of some vertex of $U$, $m: V \longrightarrow U$, where $m$ is injective—in this somewhat unusual notation, a matched edge is of the form $\{v, m(v)\}$ for $v \in V$. Then note that, for any subset $X \subseteq V$, the set $U_X = \{u \in U \mid \exists v \in X,\ \{u, v\} \in E\}$ contains at least all of the vertices $m(x)$ for $x \in X$. Since $m$ is injective, we must have $|m(X)| = |X|$ and so the set $U_X$ has cardinality at least as large as that of $X$, as desired.

The sufficient condition is harder. Assume that, for each subset $X$ of $V$, we have $|U_X| \geq |X|$. We need to prove that $G$ then has a matching of size $|V|$. We proceed by induction on the size of $V$. The statement holds vacuously for $|V| = 0$. If $V$ contains only one vertex, a matching of size $|V| = 1$ exists if there is at least one edge from that vertex to some vertex in $U$, which is exactly what the condition states. Assume then that the statement holds for all graphs with vertex sets $V$ no larger than $n - 1$ and consider a graph $G$ with vertex set $V$ of $n > 1$ vertices. We consider two cases:

- For each proper, nonempty subset $X$ of $V$, $\emptyset \subset X \subset V$, we have $|U_X| > |X|$. Then pick any vertex $v$ in $V$ and use any edge incident upon $v$ to match $v$, say to vertex $u \in U$. Then consider the induced bipartite graph on $U - \{u\}$ and $V - \{v\}$. This new graph obeys the inductive hypothesis (we had strict inequalities and now we have reduced the size of the $U_X$ for any $X$ by at most 1, so we have, at worst, equalities) and thus it has a matching of size $|V| - 1$; adding in the edge $\{v, u\}$, which cannot, by construction, share any endpoint with the inductively defined matching, yields a matching of size $|V|$ for the original graph.
- There is a nonempty proper subset $X$ of $V$ with $|U_X| = |X|$. The induced bipartite graph formed by $X$ and $U_X$ obeys the inductive hypothesis, so that it has a matching of size $|X|$. Consider then the

induced bipartite graph formed by $V - X$ and $U - U_X$; it too obeys the inductive hypothesis! (Otherwise, there would be a set $A \subseteq V - X$ with $|A| > |U_A|$ and thus such that $|A \cup X| > |U_{A \cup X}|$ in the original graph, a contradiction.) Thus this induced graph has a matching of size $|V| - |X|$, one that is entirely disjoint (no shared endpoint) from the matching obtained for $X$. The union of the two matchings is thus a matching of size $|V|$ for the original graph, as desired.