The Theory of Computation

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Addison-Wesley-Longman, 1998

Solution Manual
Chapter 2, Problem 2.22

By Exercise 2.21, a graph with 11 vertices cannot have more than $3 \cdot 11 - 6 = 27$ edges; thus its complement must have at least $(11 \cdot 10)/2 - 27 = 28$ edges. But no planar graph on 11 vertices may have more than 27 edges, so that the complement must be nonplanar.

The calculation is easy: let the number of vertices be n. Then the largest number of edges that the graph may have (and remain planar) is $3n - 6$; the complement of a graph of n vertices and $3n - 6$ edges has $n(n-1)/2 - (3n-6) = \frac{1}{2}(n^2 - 7n + 12)$ edges. In order for the complement to remain planar, it also must have at most $3n - 6$ edges, yielding the inequality $\frac{1}{2}(n^2 - 7n + 12) \leq 3n - 6$, or $n^2 - 13n + 24 \leq 0$, with roots $\frac{1}{2}(13 + \sqrt{73})$ and $\frac{1}{2}(13 - \sqrt{73})$ Since the square root of 73 is slightly larger than 8.5, our n must be less than $\frac{1}{2}(13 + 8.5)$ and an integer, that is, must be at most 10. Thus 11 is indeed the smallest value with that property, as could also have been deduced from the fact that the the number of edges of the complement was the smallest value that would violate the limit and that the number of edges of the complement grows quadratically while that of the planar graph grows only linearly.