The Theory of Computation

Bernard M.E. Moret

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Solution Manual

Chapter 2, Problem 2.20

We want to prove the equality $|R| = |E| - |V| + 2$. A non-trivial region must have at least 3 vertices and 3 edges on its boundary; each edge can be shared by at most two regions.

We exclude a graph with no edge and no vertex. The three smallest nonempty graphs otherwise all obey the equality. If the graph has 1 vertex, it has 0 edges, and 1 region, so that the equality holds. If the graph has 1 edge, then it has 1 regions; it also has 2 vertices, so the equality holds. For 2 edges, we still have 1 region, but 3 vertices; again, the equality holds.

Let us then assume that the equality holds for graphs of up to m edges, that is, we have $r = m - n + 2$ for all such graphs, where r is the number of regions and n the number of vertices. Since our basis covers cases up to $m = 2$ and $n = 3$, assume our graph already has at least 2 edges and 3 vertices. If we add an edge, we can add it in just three ways:

1. We add one vertex and one edge by placing the vertex within a region and connecting a vertex on the boundary of the region to the new vertex; the number of regions does not change, so the equality is respected.
2. We split an existing edge into two edges by placing a vertex in the "middle." The number of regions is unaffected, while the number of vertices and the number of edges both go up by 1; hence the equality is respected.
3. We add an edge between two existing vertices; in that case, these vertices must both be on or in the same region (otherwise planarity would be destroyed), so that the new edge will split that region into two. The number of edges goes up by 1, the number of regions goes up by 1, and the number of vertices remains unchanged, so that the equality is respected.

No other way is possible: adding an edge and two vertices would imply that the new edge is unattached—but we must build a connected graph; adding an edge between two existing vertices without splitting a region is not possible once the graph has at least 3 vertices—as ours does by inductive hypothesis.