The Theory of Computation

Bernard M.E. Moret

Addison-Wesley-Longman, 1998

Solution Manual

Chapter 2, Problem 2.15

To prove that the two define the same concept, we need to prove two implications. Actually, as we see below, one implication is true (the second one) and one is only "almost" true—i.e., false . . .

- If f is $\Theta(g)$, then the limit of the ratio $\frac{f}{g}$ is a positive constant.

  Our definition is a symmetric $O(\,)$. Thus there exist positive constants c_1, c_2, N_1, and N_2 such that, for all $n > N_1$, $f(n) \leq c_1 g(n)$ and, for all $n > N_2$, $g(n) \leq c_2 f(n)$. Let $N = \max(N_1, N_2)$; then the two inequalities hold for all $n > N$. By dividing both inequalities by $g(n)$ and isolating the ratio $\frac{f}{g}$, we get the inequalities

  $$\frac{1}{c_2} \leq \frac{f(n)}{g(n)} \leq c_1 \text{ for all } n > N$$

  Thus the ratio is bounded above and below by two positive constants This is not exactly the same as postulating that the limit of the ratio is a fixed positive constant; conceivably, we could have strange functions that make the ratio oscillate forever between $\frac{1}{c_2}$ and c_1, so that the limit does not exist. However, the difference is negligible in practice—and is easy to fix by replacing the ratio $\frac{f}{g}$ by $\sup \frac{f}{g}$ (or by $\inf \frac{f}{g}$), the supremum (infimum) of the ratio, thereby ensuring that the limit exists.

- If the limit of the ratio $\frac{f}{g}$ is a positive constant, then f is $\Theta(g)$ by our definition.

  We need to return to calculus and the definition of limits. If we know that the limit of the ratio is a fixed constant c, it means that, for each positive constant ϵ, there exists a positive constant N such that

  $$\left| \frac{f(n)}{g(n)} - c \right| \leq \epsilon, \text{ for all } n > N$$

or, rewriting as two inequalities,

$$c - \epsilon \le \frac{f(n)}{g(n)} \le c + \epsilon, \text{ for all } n > N$$

Comparing with our derivation for the other implication, we see that this is equivalent to our definition of $\Theta(\,)$, with $\frac{1}{c_2} = c - \epsilon$ and $c_1 = c + \epsilon$.