The Theory of Computation

Bernard M.E. Moret

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Solution Manual
Chapter 2, Problem 2.12

If 3^n were $O(2^n)$, there would exist $c, N > 0$ such that, for all $n > N$, $3^n \leq c \cdot 2^n$, or, equivalently, $\frac{3^n}{c \cdot 2^n} \leq 1$. But, for any constant $c > 0$,

$$\lim_{n\to\infty} \frac{3^n}{c \cdot 2^n} = c \cdot \lim_{n\to\infty} \left(\frac{3}{2}\right)^n$$

diverges, so that no such c and N can exist.

On the other hand, we can write $3^n = 2^{\alpha n}$ and solve for α, obtaining $\alpha = \log_2 3$. Hence, for any $\alpha \geq \log_2 3$, it is indeed the case that 3^n is $O(2^{\alpha n}$.

Our first line of reasoning is not directly affected if we replace n by n^k for $k > 1$—the ratio simply diverges a little faster, since it now becomes $\frac{3}{2}^{n^k}$. Thus we can state that 3^{n^k} is not $O(2^{n^k}$ for $k > 1$.

Our second line of reasoning is equally unaffected; we have

$$2^{\log_3 2 \cdot n^k} = \left(2^{\log_2 3}\right)^{n^k} = 3^{n^k}$$

so that the analogous statement holds: for any $\alpha \geq \log_2 3$, it is indeed the case that 3^{n^k} is $O(2^{\alpha n^k}$.